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Split bases and multiplicity separations in symmetric group transformation coefficients

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Abstract. We consider matrices transforming between the standard Young–Yamanouchi basis of the symmetric group S_n and bases adapted to the product subgroups $S_{n-b} \times S_b$ (the split basis). We derive closed formulae for transformation coefficients for $b = 3$, which includes the first cases when a choice of multiplicity separation is required. We discuss considerations which can be applied to obtain a simple form for the multiplicity separation. We show that the combinatorial and algebraic structure of the Littlewood–Richardson rule, also known as the Biedenharn–Louck pattern calculus, does not assist with finding a simple multiplicity separation.

1. Introduction

Transformation coefficients between various bases of symmetric groups have many uses, both directly and with various transformation coefficients of the unitary groups via the Schur–Weyl duality (Elliott *et al* 1953, Kramer 1968, Vanagas 1971, Haase and Butler 1984a, b). A key outstanding problem is to resolve multiplicity separations (see Butler 1981 p 25) in a systematic manner for the symmetric and the unitary groups.

Biedenharn (1963), Baird and Biedenharn (1963–5) and Biedenharn and Louck (e.g. 1972) have argued that the pattern calculus (a variant of the Littlewood–Richardson rule) should provide such an algorithm for labelling the multiplicity terms. In particular Biedenharn, Louck and Baird are proponents of techniques dependent on the use of the standard unitary basis ($U_n \supset U_{n-1} \supset \cdots \supset U_1$). This leads them to propose the product terms be labelled by basis kets of the ket-space, the (unitary) Gel'fand pattern. Butler (1975) argues that the product terms may be able to be labelled by Young–Yamanouchi symbols of symmetric groups. He mentions that these alternative labelling schemes may coincide due to the combinatorial similarities between unitary and symmetric groups.

In essence the problem is whether there are combinatoric labels that give a canonical separation of the multiplicity. We analyse the simplest multiplicity class. It arises in $S_n \supset S_{n-3} \times S_3$. We prove that the 'null space pattern' of Biedenharn *et al* does not apply to multiplicities.

In this paper we focus on the transformation between the standard Young–Yamanouchi basis and a second basis for which we introduce the term *split basis* and denote it as the S_n–$S_{a,b}$-basis. It is adapted to a direct product subgroup $S_a \times S_b$. Elliott *et al* (1953) introduced this basis, as it was a big step in evaluating coefficients of fractional parentage (cfp) for nuclear shell models. Although calculations of cfp have been undertaken for many

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years, there is still a need for more efficient methods. Indeed in cases of multiplicity, no systematic method exists.

Studies of the split-to-standard transformation coefficients have yielded several general numerical techniques for calculating the transformation coefficients (Horie 1964, Kaplan 1961, 1975, Chen *et al* 1983, Pan and Chen 1993) as well as closed formulae for particular cases (Kaplan 1975, Suryanarayana and Rao 1982). The method of Pan and Chen (1993) is particularly useful since it provides q-dependent algebraic solutions for the Hecke algebra, $H_n(q)$. Suryanarayana and Rao (1982) extend a formula obtained by Horie (1964) to give a closed formula for irreducible representations (irreps) of S_b with at most two columns. These results include multidimensional irreps of S_b but no multiplicities in the product of the irreps of S_a and S_b. Both Chen *et al* and Pan and Chen present calculations for at least one case with such a multiplicity. However, they choose different multiplicity separations.

The first product multiplicity in the symmetric groups occurs in S_6, where an irrep product in $S_3 \times S_3$ contains an irrep twice. This case is included in a general solution that we derive for S_n–$S_{3,3}$-bases. We use the linear equation method of Pan and Chen (1993) to derive the formulae for the coefficients associated with shapes remaining after removing the S_3 irrep from the S_n irrep. We give the general solution before any choice of multiplicity separation is made. We present a coherent framework of considerations which can be applied to obtain simple forms for the separation and thus for the transformation coefficients. We find a class of particularly simple separations.

None of the possible separations satisfy the Biedenharn–Louck conjecture that the multiplicity can be tied down using a null space structure arising from the pattern calculus (equivalently the Littlewood–Richardson rule).

In section 2 we define the standard basis and the split bases. We provide the background for constructing the representation matrices in the standard basis. Various orderings of the basis tableaux are given.

In section 3 we derive algebraic formulae, using an algorithm based on the linear equation approach of Pan and Chen (1993). We first re-derive Kaplan's (1975) formula for removing two boxes from the right. We then consider the cases relating to the removal of three boxes from the right; that is, the cases defined by the relative positions of the last three boxes in the basis tableaux. For small hook lengths these cases are related.

In section 4 we present six considerations which are used to identify which sets of transformation coefficients are to be considered simple. We demonstrate that one set of separations is particularly simple.

We summarize and discuss our results in section 5.

2. The split bases for symmetric groups

In this section we outline the necessary background on symmetric group bases and on the ordering of basis tableaux. We will work with the standard orthogonal basis for symmetric group representations (see, for example, Rutherford (1948), Hamermesh (1962), Young (1977)) and will call it the S_n-basis.

The irreducible representations of S_n may be labelled by partitions $[\lambda]$ of n. A partition of n into i parts may be written as $[\lambda_1, \lambda_2, \ldots, \lambda_i]$ such that $\sum_{j=1}^{i} \lambda_j = n$ and the λ_j are weakly decreasing. By forming a left-justified array with λ_j boxes on the jth row and with the kth row below the $(k-1)$th row, we obtain a Ferrers diagram. Young tableaux are generated by filling the Ferrers diagram with the numbers $1, \ldots, n$ such that each number appears exactly once and the numbers strictly increase across rows and down columns. The number of Young tableaux for a given partition of n is equal to the dimension of an irrep of S_n, and each basis vector can be associated with a unique tableau.

Since any permutation can be generated as a product of adjacent transpositions, it suffices to consider just the representation matrices for such transpositions. These matrices can be calculated simply using the tableau parameter of *hook length* (sometimes known as the *axial distance*) (see Rutherford (1948, pp 41–9); also Young (1977 [VI, pp 217, 218])). The hook length between the box containing i, at (x_i, y_i), and the box containing j, at (x_j, y_j), is defined as $\tau_{ij} = (x_j - x_i) - (y_j - y_i)$. We write the S_n-basis representation matrix for the adjacent transposition $(k-1, k)$ in the representation λ as $M^{\lambda}((k-1, k))$. For a more general permutation, σ, we write $M^{\lambda}(\sigma)$.

An alternative basis for S_n, which we call the *split basis* and denote it as the S_n–$S_{n-b,b}$-basis, has basis functions adapted to S_n and to the direct product subgroup $S_{n-b} \times S_b$. For each factor group we choose the S_{n-b}-basis and S_b-basis, respectively. When $b = 1$, the S_n–$S_{n-b,b}$-basis is the S_n-basis. One can label the basis vectors of the split basis by a pair of Young tableaux, the first with $a = n - b$ boxes and the second with b boxes. These tableaux determine the representation matrices of most adjacent transpositions in the split bases as with the standard basis. If the adjacent transposition is in S_a the hook lengths in the first tableau are used. If the adjacent transposition is in S_b the hook lengths in the second tableau are used. The bridging transposition $(n-b, n-b+1)$ cannot be calculated in this manner, but can be found by using the split-to-standard transformation matrices.

To define different bases, and to order them, we make use of a combinatorial technique known as *jeu de taquin*. *Jeu de taquin*, or simply *jeu*, is due to Schützenberger (1963) and is a procedure for removing a box from a Young tableau and then filling the hole created by this removal so that the ultimate result is itself a Young tableau of standard shape. *Jeu* can be described as follows:

> Remove a box from the Young tableau. Examine the number in the box to the right and the number in the box below the position of the removed box. Select the number that is smallest and move the box containing it into the empty space. While there are still boxes to the right in the same row as the hole, or lower in the same column as the hole, repeat this procedure.

Authors have made different order choices on the set of Young tableaux. One popular ordering is *last letter order*. This is the ordering used by Chen *et al* (1983), who call it decreasing page order. When we say that a is lower in a tableau than b, we mean a is either in a lower row, or in the same row and to the right of b. Given two tableaux, T and U, T precedes U in last letter order if the last letter of disagreement between the two is lower in T than in U. The complementary ordering is *first letter order*. Given two tableaux, T and U, T precedes U in first letter order if the first letter of disagreement between T and U is lower in U than it is in T (Young 1977 [IV, p 258]). We will also make use of a third kind of ordering, one dependent on the form of the split basis. Chen *et al* use this ordering for the split basis without describing it in the manner we do now. Pan and Chen (1993) use a similar ordering but are not completely consistent in their tables.

To define this third ordering we first establish a correspondence between a tableau of shape λ in the S_n-basis and a pair of tableaux of shape (α, β) in the split S_n–$S_{a,b}$-basis. Given a tableau of shape λ in the S_n-basis, remove the tableau, α, consisting of the boxes containing the first a labels. This tableau is the first in the $S_a \times S_b$ pair. Then apply *jeu* to the remaining b boxes to make a tableau of standard shape, β. This tableau β is the second in the $S_a \times S_b$ pair. This correspondence is many-to-one so that a single $\alpha\beta$ pair can have many tableaux of shape λ in the S_n-basis that map to it. Order the tableau pairs (α, β) in the S_n–$S_{a,b}$-basis in the following manner. First adopt an order on partitions such that $\lambda < \rho$ if

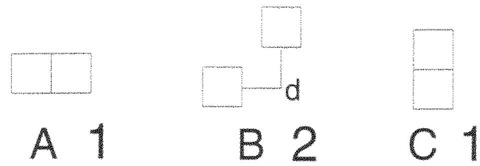

Figure 1. The skew shapes remaining after the removal of $(n-2)$ boxes from the left.

$\lambda_i = \rho_i$ for $1 \leqslant i \leqslant k$ and $\lambda_k > \rho_k$. Now order the (α, β) pairs by the α, using the partition order defined above if the shapes in two pairs are different, and by first letter order if the shapes are the same. Then for pairs in which the first components are identical, order by the second components, again first using partition order, and then first letter order. When a pair occurs more than once, the situation known as product multiplicity, order those pairs according to the first letter ordered list of standard tableaux from which they came. This gives a unique ordering for any split basis.

3. Formulae for removing three boxes

Consider a general transformation from the S_n-basis to the S_n–$S_{n-b,b}$-basis. We can break the transformation matrix into cases associated with the shape of the tableau obtained by removing $n-b$ blocks from the basis tableau of the S_n-basis. This produces two shapes, α and λ/α. The first is standard and associated with irreps of the S_{n-b} subgroup; the second shape is skew and associated with (non-standard) irreps of the S_b subgroup. Standardizing the second shape using *jeu* gives the second irrep of the pair labelling the split basis, β. Permutations cannot change the shape of the first irrep of the pair. Indeed as Chen *et al* point out, Schur's lemma furthermore implies that the transformation coefficient is the same for each basis vector of this first irrep. Thus we can split the transformation coefficient matrix into blocks. The blocks are of the dimension, $|\lambda/\alpha|$, of the skew shape remaining after removing the first $n-b$ boxes. The basis vectors of the split basis associated with the basis vectors of the S_n-basis are given by the process described in section 2.

Each of the dimension one irreps, $[b]$ and $[1^b]$, will always give rise to a single 1×1 block with a simple phase freedom.

Let us first consider the two-box example of Kaplan (1975). There are three cases (see figure 1). The transformation coefficients of the first and last are phases ± 1. The interesting case is $\mathcal{B}$.

We begin by ordering the basis tableaux according to the prescription of section 2. Label them alphabetically (upper-case) according to this order, with the split basis tableaux labels having a prime attached. Consider then the two-way expansions of the entries on the diagonal of the 2×2 transformation matrix,

$$\begin{aligned}\langle A'|(n-1, n)|A\rangle &= \langle A'|A\rangle \\ &= -\frac{1}{d}\langle A'|A\rangle + \frac{\sqrt{d^2-1}}{d}\langle A'|B\rangle\end{aligned}$$

where d is the absolute value of the axial distance from the box containing $n-1$ to the box containing n, in the tableau labelled A. With the same d,

$$\begin{aligned}\langle B'|(n-1, n)|B\rangle &= -\langle B'|B\rangle \\ &= \frac{1}{d}\langle B'|A\rangle + \frac{\sqrt{d^2-1}}{d}\langle B'|B\rangle.\end{aligned}$$

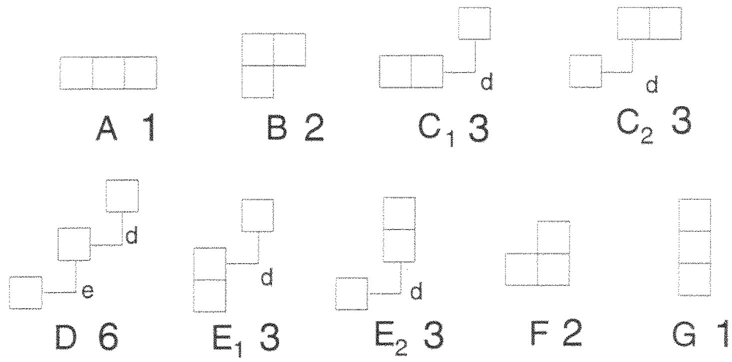

Figure 2. The skew shapes remaining after the removal of $(n-3)$ boxes from the left.

Applying normality gives, with $\theta = \pm 1$, a choice of phase,

$$\begin{pmatrix} \theta\sqrt{\frac{d-1}{2d}} & \theta\sqrt{\frac{d+1}{2d}} \\ -\theta\sqrt{\frac{d+1}{2d}} & \theta\sqrt{\frac{d-1}{2d}} \end{pmatrix}. \tag{3.1}$$

Notice that our ordering differs from that of Kaplan (1975, equation (2.65), p 51).

Now let us proceed to removing three boxes. The cases are listed in figure 2, which gives the skew shape and the relevant dimension for the associated transformation matrix.

We used the package MAPLE to implement a formalized algorithm based on the linear equation method of Pan and Chen (1993). The formulae depend on the various hook lengths, $d, e, f = d + e, d_+ = d + 1$ as well as augmented hook lengths, $h_\pm = h \pm 1$. Two phases, θ and ϕ, occur in the multiplicity free cases. The general formulae follow.

Case $\mathcal{A}$. Completely symmetric, θ.

Case $\mathcal{B}$.

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}. \tag{3.2}$$

Case $\mathcal{C}_1$.

$$\begin{pmatrix} \theta\sqrt{\frac{d_-}{3d_+}} & \theta\sqrt{\frac{d_-d_{++}}{3dd_+}} & \theta\sqrt{\frac{d_{++}}{3d}} \\ -\phi\sqrt{\frac{d_{++}}{6d_+}} & -\phi\frac{d_{++}}{\sqrt{6dd_+}} & \phi\sqrt{\frac{2d_-}{3d}} \\ -\phi\sqrt{\frac{d_{++}}{2d_+}} & \phi\sqrt{\frac{d}{2d_+}} & 0 \end{pmatrix}. \tag{3.3}$$

Case $\mathcal{C}_2$.

$$\begin{pmatrix} \theta\sqrt{\frac{d_-}{3d_+}} & \theta\sqrt{\frac{d_-d_{++}}{3dd_+}} & \theta\sqrt{\frac{d_{++}}{3d}} \\ -\phi\sqrt{\frac{2d_{++}}{3d_+}} & \phi\frac{d_-}{\sqrt{6d_+d}} & \phi\sqrt{\frac{d_-}{6d}} \\ 0 & -\phi\sqrt{\frac{d_+}{2d}} & \phi\sqrt{\frac{d_-}{2d}} \end{pmatrix}. \tag{3.4}$$

Case $\mathcal{D}$. The solutions for the four central rows, corresponding to the multiplicity problem, are given in equations (3.10).

$$\begin{pmatrix} \theta\sqrt{\frac{d_-e_-f_-}{6def}} & \theta\sqrt{\frac{d_-e_+f_-}{6def}} & \theta\sqrt{\frac{d_+e_-f_-}{6def}} & \theta\sqrt{\frac{d_-e_+f_+}{6def}} & \theta\sqrt{\frac{d_+e_-f_+}{6def}} & \theta\sqrt{\frac{d_+e_+f_+}{6def}} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \phi\sqrt{\frac{d_+e_+f_+}{6def}} & -\phi\sqrt{\frac{d_+e_-f_+}{6def}} & -\phi\sqrt{\frac{d_-e_+f_+}{6def}} & \phi\sqrt{\frac{d_+e_-f_-}{6def}} & \phi\sqrt{\frac{d_-e_+f_-}{6def}} & -\phi\sqrt{\frac{d_-e_-f_-}{6def}} \end{pmatrix}. \tag{3.5}$$

Case $\mathcal{E}_1$.

$$\begin{pmatrix} \theta\sqrt{\frac{d_-}{2d}} & \theta\sqrt{\frac{d_+}{2d}} & 0 \\ -\theta\sqrt{\frac{d_-}{6d}} & \theta\frac{d_-}{\sqrt{6d_+d}} & \theta\sqrt{\frac{2d_{++}}{3d_+}} \\ \phi\sqrt{\frac{d_{++}}{3d}} & -\phi\sqrt{\frac{d_-d_{++}}{3d_+d}} & \phi\sqrt{\frac{d_-}{3d_+}} \end{pmatrix}. \tag{3.6}$$

Case $\mathcal{E}_2$.

$$\begin{pmatrix} 0 & -\theta\sqrt{\frac{d}{2d_+}} & -\theta\sqrt{\frac{d_{++}}{2d_+}} \\ -\theta\sqrt{\frac{2d_-}{3d}} & -\theta\frac{d_{++}}{\sqrt{6d_+d}} & \theta\sqrt{\frac{d_{++}}{6d_+}} \\ \phi\sqrt{\frac{d_{++}}{3d}} & -\phi\sqrt{\frac{d_-d_{++}}{3d_+}} & \phi\sqrt{\frac{d_-}{3d_+}} \end{pmatrix}. \tag{3.7}$$

Case $\mathcal{F}$.

$$\begin{pmatrix} \frac{\theta}{2} & \frac{\sqrt{3}\theta}{2} \\ \frac{\sqrt{3}\theta}{2} & -\frac{\theta}{2} \end{pmatrix}. \tag{3.8}$$

Case $\mathcal{G}$. Completely anti-symmetric, θ.

We now return to consideration of the four central rows of case $\mathcal{D}$, the multiplicity case. The system of equations includes three orthonormality equations, thus three independent phase choices exist, θ, ϕ, ψ. There is one free factor governing multiplicity separation. We express all coefficients in terms of two judiciously chosen variables, x and y and their ratio, $r = y/x$. We let

$$x = 1/\sqrt{6def(2de + d - e + 1)(1 + 3d_+d_-e_+e_-f_+f_-r^2)}. \tag{3.9}$$

The solutions, where the a_{ij} are shown in equation (3.5), are

$$a_{11} = \sqrt{d_-/d_+}a_{13} \qquad a_{12} = \sqrt{f_-/f_+}a_{14} \qquad a_{15} = \sqrt{e_-/e_+}a_{16}$$

$$a_{13} = -d_+\sqrt{f_-}[\theta e_{++}x + 3\psi d_-e_+e_-f_+y]$$

$$a_{14} = \sqrt{d_+d_-e_+e_-f_+}[2\theta x - 3\psi f_+f_-y]$$

$$a_{16} = \sqrt{e_+e_-f_+}[-\theta d_{--}x + 3\psi d_+d_-e_+f_-y]$$

$$
\begin{aligned}
&a_{21} = \sqrt{d_-/d_+}a_{23} \qquad a_{22} = \sqrt{f_-/f_+}a_{24} \qquad a_{25} = \sqrt{e_-/e_+}a_{26} \\
&a_{23} = -\sqrt{3d_+d_-e_+e_-f_+}[x + \phi d_+e_{++}f_-y] \\
&a_{24} = f_+\sqrt{3f_-}[-x + 2\phi d_+d_-e_+e_-y] \\
&a_{26} = e_+\sqrt{3d_+d_-f_-}[x - \phi d_{--}e_-f_+y] \\
&a_{31} = -\sqrt{d_+/d_-}a_{33} \qquad a_{32} = -\sqrt{f_+/f_-}a_{34} \qquad a_{35} = -\sqrt{e_+/e_-}a_{36} \\
&a_{33} = d_-\sqrt{3f_-}[-\theta ex + \psi d_+e_+e_-f_+y] \\
&a_{34} = \psi(2de + d - e + 1)f_-\sqrt{3d_+d_-e_+e_-f_+}y \\
&a_{36} = \sqrt{3e_+e_-f_+}[\theta dx + \psi d_+d_-e_-f_-y] \\
&a_{41} = -\sqrt{d_+/d_-}a_{43} \qquad a_{42} = -\sqrt{f_+/f_-}a_{44} \qquad a_{45} = -\sqrt{e_+/e_-}a_{46} \\
&a_{43} = \sqrt{d_+d_-e_+e_-f_+}[x - 3\phi d_-ef_-y] \\
&a_{44} = (2de + d - e + 1)\sqrt{f_-}x \\
&a_{46} = e_-\sqrt{d_+d_-f_-}[x + 3\phi de_+f_+y].
\end{aligned} \tag{3.10}
$$

4. Choices of phase and multiplicity separation

We want to find the simplest and most natural form for these symmetric group transformation coefficients. There are a number of important considerations in this regard.

(I) The transformation coefficients should be chosen to be real if possible. (The expressions in section 3 assume this reality choice, as does the form of the algorithm used.)

(II) The general formulae obtained depend only upon the hook lengths, and are independent of n. Thus phases and the multiplicity separation should be chosen independent of n.

(III) When either of the hook lengths d or e is unity, the multiplicity is lifted. The expression for the multiplicity-two coefficients must reduce to the multiplicity free solutions.

(IV) The multiplicity separation is to be chosen so that a maximal number of zero coefficients is obtained.

(V) It is desirable to have the coefficients expressible as a single surd of the form $a\sqrt{b}/c$, with a, b, c integers.

(VI) We ask that the prime numbers which occur in the surds are as small as possible (relative to d and e).

Butler (1981 p 241) has raised some these considerations before. Butler (1981) proved that some transformation coefficients, in particular the set of T–D_2–3jm, satisfy neither (I) nor (V). Butler and Ford (1979) also proved that (IV) and (VI) were equivalent for octahedral 6j and derived a table of 6j which had both smaller prime numbers and more zeros than Griffith (1962).

In section 3 we expressed the coefficients in a form so that the above considerations may be easily taken into account.

Particular choices of x (and y) will cause various pairs of coefficients to vanish. There are 12 such zero conditions, which are distinct, so that consideration (IV) gives a maximum of two zeros. All zero conditions satisfy (V). Indeed only if $r = y/x$ is a rational function of d and e do all the coefficients satisfy (V) Pan and Chen (1993). Choose a separation that satisfies (V), but not (IV).

Zero conditions on coefficients which differ only in the labelling of the multiplicity,

(a_{11}, a_{13}) and (a_{21}, a_{23}) for example, give sets of coefficients differing only in the multiplicity label. This symmetry allows us to retain just the six zero conditions associated with first and third row coefficients.

Of the six distinct zero conditions we now ask which satisfy (VI). We see that the largest potentially prime factors in the transformation coefficients appear in x, $\sqrt{2de + d - e + 1}$ and $\sqrt{1 + 3d_+d_-e_+e_-f_+f_-r^2}$. The former is of order two in hook lengths, but the latter, which can be written

$$\sqrt{1 + 3(d^2 - 1)(e^2 - 1)(f^2 - 1)r^2} \tag{4.1}$$

is potentially troublesome. However, for all of the zero conditions we can factorize this expression into terms of no greater than order two in d and e. For example, setting $a_{35} = a_{36} = 0$ gives $r = -\theta d/(\psi d_+d_-e_-f_-)$, so that (4.1) reduces to

$$\sqrt{(2d^2 + 2de + 2d + e - 1)(2de + d - e + 1)/(d_+d_-e_-f_-)}. \tag{4.2}$$

To distinguish between the six zero conditions let us use (III) to look at the restriction of hook lengths so that the multiplicity is lifted. This occurs when either d or e is unity. However, because of the d_-e_- term in (3.9), the dependence of x on r is lost. Putting this degeneracy aside and setting $d = e = 1$ in (3.10) we obtain the submatrix

$$\begin{pmatrix} 0 & 0 & -\theta & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{4.3}$$

From figure 2 we see that those coefficients reduce to those for cases $\mathcal{B}$ and $\mathcal{F}$, given in (3.2) and (3.8) respectively. The second and fourth columns relate to $\mathcal{F}$ and the third and fifth to $\mathcal{B}$. The first and last columns correspond to the irreps $[1^3]$ and [3], not [2 1], and so must be zero. Four of the six zero conditions have zeros in positions that either conflict directly with (4.3), or with the corresponding matrices when only one of d or e is unity.

The six considerations for simplicity do not strongly distinguish between the two remaining solutions: (a_{12}, a_{14}) and (a_{32}, a_{34}). The magnitudes of coefficients for the former condition are dependent upon phase choices. Choosing $\psi\phi\theta = 1$ gives the simplest magnitudes, and the resulting coefficients are related to coefficients of the (a_{32}, a_{34}) solution. Thus we conclude that the solution associated with this (a_{32}, a_{34}) zero condition is the simplest and thus best choice of multiplicity separation. Defining $\alpha = \sqrt{6def(2de + e - d + 1)}$

$$\begin{aligned} a_{13} &= -\theta d_+e_{++}\sqrt{f_-}/\alpha & a_{23} &= -\sqrt{3d_+d_-e_+e_-f_+}/\alpha \\ a_{14} &= 2\theta\sqrt{d_+d_-e_+e_-f_+}/\alpha & a_{24} &= -f_+\sqrt{3f_-}/\alpha \\ a_{16} &= -\theta d_{--}\sqrt{e_+e_-f_+}/\alpha & a_{26} &= e_+\sqrt{3d_+d_-f_-}/\alpha \\ a_{33} &= -\theta ed_-\sqrt{3f_-}/\alpha & a_{43} &= \sqrt{d_+d_-e_+e_-f_+}/\alpha \\ a_{34} &= 0 & a_{44} &= (2de + d - e + 1)\sqrt{f_-}/\alpha \\ a_{36} &= \theta d\sqrt{3e_+e_-f_+}/\alpha & a_{46} &= e_-\sqrt{d_+d_-f_-}/\alpha. \end{aligned} \tag{4.4}$$

This is the choice of multiplicity separation made by Chen *et al* (1983) for the $d = e = 2$ case. The choice of Pan and Chen (1993), for the Hecke algebras is different. They introduce a symmetry requirement on their S_6 separation which is inconsistent with our consideration (IV). Both published solutions for S_6 satisfy (VI), having the largest prime in a surd as 5. (There are a few errors in the tables of Chen *et al*: table I.1 of phase factors Λ^{μ}_{m}, the sixth factor for [3 1 1] has the wrong sign; table II.16 {1122} with 1 has the wrong sign;

table II.17 {2111} with 8 should be 5 rather than 0; table II.22 {2132} with 16 should be 2 rather than 0.)

Let us now use the considerations (II), (III) and (IV) to examine phase choices for all solutions.

For $d = 1$ cases $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{E}_1$, $\mathcal{E}_2$ collapse to cases $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{G}$. This gives the following relations between the phases.

$$\begin{array}{llll} \theta_{C_1} = \theta_A & \phi_{C_1} = -\theta_F & \theta_{C_2} = \theta_A & \phi_{C_2} = -\theta_B \\ \theta_{E_1} = \theta_B & \phi_{E_1} = \theta_G & \theta_{E_2} = -\theta_F & \phi_{E_2} = \theta_G. \end{array} \tag{4.5}$$

Setting $d = e = 1$ in the first and last rows of case $\mathcal{D}$, as given in section 3, we find that

$$\theta_D = \theta_A \qquad \phi_D = \theta_G. \tag{4.6}$$

We thus need to choose four phases. We choose,

$$\theta_A = \theta_B = -\theta_F = \theta_G = 1 \tag{4.7}$$

where we use the negative θ_F so that $\theta_G = \phi_{C_2}$ in (4.5).

5. Summary

The investigation in this paper was motivated by four factors. First, the split basis itself is not completely understood, especially in the sense that the bridging transposition in the split basis cannot be calculated as directly as other transpositions. Second, a general formula for the split-to-standard transformation coefficients is not available and we wished to extend the two box formulae of Kaplan (1975) to the case of three boxes. Third, we observed that Pan and Chen (1993) and Chen *et al* (1983) make different multiplicity choices for the first situation where such a choice is necessary. The fourth, and main motivation, was the desire to find out if the Littlewood–Richardson rule was enough to give a canonical multiplicity separation.

We have presented the explicit formula for the transformation coefficients between the standard (Young–Yamanouchi) basis and the split basis corresponding to the removal of three boxes. The results are presented in terms of the nine cases distinguished by the skew shape remaining after removing $n-3$ boxes from the left. We have obtained the general multiplicity two solutions for the S_n–$S_{n-3,3}$-basis. We have discussed six considerations so as to compare transformation coefficients for simplicity. The simpler separations were found to correspond to one of 12 zero conditions. These occurred in pairs linked by relabelling the multiplicity. Two of these six pairs matched the solutions of degenerate cases. We fixed upon a solution where phases and magnitudes had their simplest numerical form.

In our complete solution to this multiplicity problem we proved that the Littlewood–Richardson rule (the pattern calculus of Biedenharn and Louck (1981)) does not provide a specific separation, and that the 'null space pattern' envisaged by Biedenharn *et al* does not exist. When no multiplicities exist the Littlewood–Richardson rule gives the pattern relations between the split and standard bases. The pattern relations implicit in the combinatorial structure of the Littlewood–Richardson rule do not determine a canonical set of basis functions for the bases labelled by multiplicity labels. Rather, a choice must be made using criteria beyond the Littlewood–Richardson rule.

We reviewed and extended considerations for making the choice of the multiplicity separation. We showed that in this $S_n \rightarrow S_n$–$S_{n,n-3}$-basis transformation, these six considerations could be simultaneously satisfied. The next steps in a search for a combinatorial recipe for a multiplicity separation could be to look at other multiplicity-two cases and the first multiplicity-three case. The multiplicity-three case first occurs in the decomposition of [4 3 2 1] of S_{10} into [321] $\times$ [31] of $S_6 \times S_4$.

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